Elliptic Gaudin models and elliptic KZ Equations

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Abstract

The Gaudin models based on the face-type elliptic quantum groups and the XYZ Gaudin models are studied. The Gaudin model Hamiltonians are constructed and are diagonalized by using the algebraic Bethe ansatz method. The corresponding face-type Knizhnik-Zamolodchikov equations and their solutions are given.

I Introduction

In [1], Gaudin proposed a quantum integrable model describing N spin 1/2 particles with long-range interactions. The Gaudin type models played an important role in establishing the integrability of the Seiberg-Witten theory [2, 3], and were used as a testing ground for ideas such as the functional Bethe ansatz and general procedure of separation of variables [4, 5, 6]. They also have direct applications in condensed matter physics.

The algebra associated with the model proposed by Gaudin is the Lie algebra $\widehat{sl}_2$. Gaudin's work was later generalized by several authors [7, 8, 9] and integrable Gaudin models based on other Lie algebras were constructed. The XYZ Gaudin model was counstructed and solved in [5] by means of the algebraic Bethe ansatz. The boundary Gaudin magnet associated with the spin 1/2 representation of $\widehat{sl}_2$ was investigated by Hikami [10] using Sklyanin's boundary quantum inverse scattering method.

The Knizhnik-Zamolodchikov (KZ) equations were first proposed as a set of differential equations satisfied by correlation functions of the Wess-Zumino-Witten models [11]. The relation between the Gaudin magnets and the KZ equations has been studied in many papers [12, 13, 14, 15]. In [14] and [10], Hikami gave an integral representation for solutions of the KZ equations by using the results of the periodic and boundary Gaudin models.

In the present paper, we construct elliptic Gaudin models based on the face-type elliptic quantum group $E_{\tau,\eta}(\widehat{sl}_2)$ [16] and boundary elliptic quantum group $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ [17] and the boundary XYZ Gaudin models. We diagonalize them by means of the algebraic Bethe ansatz method. Moreover we construct the face-type elliptic KZ equations and their solutions using the off-shell Bethe ansatz equations of the face-type elliptic Gaudin models.

This paper is organized as follows. In section 2, we review the elliptic quantum group $E_{\tau,\eta}(\widehat{sl}_2)$ and the boundary elliptic quantum group $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$. Then in section

3, we construct the corresponding elliptic quantum Gaudin models. Solutions of the KZ equations based on these Gaudin magnets are also given. In section 4, we give the Hamiltonians, eigenvalues and Bethe ansatz equations of the boundary XYZ Gaudin model. In the last section, we present some discussions.

II Elliptic quantum groups and Bethe ansatz

In this section we review the elliptic quantum group $E_{\tau,\eta}(\widehat{sl}_2)$ [16, 18] and the boundary elliptic quantum group $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ [17]. These elliptic quantum groups are algebraic structures underlying the (boundary) dynamical Yang-Baxter equation in statistical mechanics and the KZ equations on a torus.

II.1 $E_{\tau,\eta}(\widehat{sl_2})$ and its Bethe ansatz

Let h be the generator of an one-dimensional commutative Lie algebra, and $a(\lambda,w)$, $b(\lambda,w)$, $c(\lambda,w)$, $d(\lambda,w)$ be the elements of the matrix $L(\lambda,w)$. Then $E_{\tau,\eta}(\widehat{sl}_2)$ is generated by meromorphic functions of the variable h and the elements of $L(\lambda,w)$ with non-commutative entries, subject to the dynamical Yang-Baxter equation (YBE) [16]

$$\begin{aligned} & R^{(12)}(\lambda-2\eta h,w_{12})L^{(1)}(\lambda,w_1)L^{(2)}(\lambda-2\eta h^{(1)},w_2) \\ = \; & L^{(2)}(\lambda,w_2)L^{(1)}(\lambda-2\eta h^{(2)},w_1)R^{(12)}(\lambda,w_{12}). \end{aligned} \tag{II.1}$$

Here $w_{ij}=w_i-w_j$ with w_i being spectral parameters, η is the crossing parameter; $L^{(i)}(\lambda-2\eta h^{(j)},w_i)$ acts on the i-th auxiliary space and $h^{(j)}$ on the j -th auxiliary space $V=C^2$ by a diagonal matrix $diag(1,-1)$. $R^{(12)}(\lambda-2\eta h,w_{12})$ acts on the space $C^2\otimes C^2$ with h acting on the quantum space. The R-matrix is given by

$$\begin{aligned} R(\lambda,w) \;=\; & E_{0,0}\otimes E_{0,0}+E_{1,1}\otimes E_{1,1}+\alpha(\lambda,w)E_{0,0}\otimes E_{1,1}+\alpha(-\lambda,w)E_{1,1}\otimes E_{0,0} \\ & +\beta(\lambda,w)E_{0,1}\otimes E_{1,0}+\beta(-\lambda,w)E_{1,0}\otimes E_{0,1} \end{aligned} \tag{II.2}$$

with

$$\begin{aligned} & \alpha(\lambda,w)=\frac{\theta(w)\theta(\lambda+2\eta)}{\theta(w-2\eta)\theta(\lambda)}, \quad \beta(\lambda,w)=\frac{\theta(\lambda+w)\theta(2\eta)}{\theta(w-2\eta)\theta(\lambda)}, \\ & \theta(\lambda)\equiv\theta(\lambda,\tau)=-\sum_{j\in Z}e^{\pi i(j+1/2)^2\tau+2\pi i(j+1/2)(\lambda+1/2)}. \end{aligned}$$

The commutation relations between the generators of $E_{\tau,\eta}(\widehat{sl}_2)$ are defined by eq.(II.1), and the commutation relations of the generators with some functions $f(\lambda,h)$, $g(\lambda,h)$ are [16]

$$\begin{aligned} f(\lambda,h)f(\lambda,h)=g(\lambda,h)f(\lambda,h), &\quad f(\lambda-2\eta,h)a(\lambda,w)=a(\lambda,w)f(\lambda,h), \\ f(\lambda+2\eta,h)d(\lambda,w)=d(\lambda,w)f(\lambda,h), &\quad f(\lambda+2\eta,h+2)b(\lambda,w)=b(\lambda,w)f(\lambda,h), \\ f(\lambda-2\eta,h-2)c(\lambda,w)=c(\lambda,w)f(\lambda,h). & \end{aligned} \tag{II.3}$$

For an even number $\Lambda \geq 0$ and a complex number z, we can define an evaluation module $V_\Lambda(z)$ of $E_{\tau,\eta}(\widehat{sl}_2)$. Let e_k, $k \in Z_{\geq 0}$ be a set of bases of $V_\Lambda(z)$. The action is defined by

$$\begin{aligned}
&f(h)e_k = f(\Lambda - 2k)e_k, \quad a(\lambda, w)e_k = g(a, \lambda, w, k)e_k,\\
&b(\lambda, w)e_k = g(b, \lambda, w, k)e_{k+1}, \quad d(\lambda, w)e_k = g(d, \lambda, w, k)e_k,\\
&c(\lambda, w)e_k = g(c, \lambda, w, k)e_{k-1},
\end{aligned} \tag{II.4}$$

with

$$\begin{aligned}
g(a, \lambda, w, k) &= \frac{\theta(z - w + (\Lambda + 1 - 2k)\eta)\theta(\lambda + 2k\eta)}{\theta(z - w + (\Lambda + 1)\eta)\theta(\lambda)},\\
g(b, \lambda, w, k) &= -\frac{\theta(-\lambda + z - w + (\Lambda - 1 - 2k)\eta)\theta(2\eta)}{\theta(z - w + (\Lambda + 1)\eta)\theta(\lambda)},\\
g(c, \lambda, w, k) &= -\frac{\theta(-\lambda - z + w + (\Lambda + 1 - 2k)\eta)\theta(2(\Lambda + 1 - k)\eta)\theta(2k\eta)}{\theta(z - w + (\Lambda + 1)\eta)\theta(\lambda)\theta(2\eta)},\\
g(d, \lambda, w, k) &= \frac{\theta(z - w + (-\Lambda + 1 + 2k)\eta)\theta(\lambda - 2(\Lambda - k)\eta)}{\theta(z - w + (\Lambda + 1)\eta)\theta(\lambda)}.
\end{aligned} \tag{II.5}$$

For any finite-dimensional module $W = V_\Lambda$ of $E_{\tau,\eta}(\widehat{sl}_2)$, the transfer matrix can be defined by

$$t(\lambda, w) \equiv tr_0 L(\lambda, w)) = a(\lambda, w) + d(\lambda, w). \tag{II.6}$$

Then $t(\lambda, w)$ preserves the space $H \equiv Fun(W)[0]$ of functions with values in the zero weight space and commutes pairwise on H, i.e. $t(\lambda, w)t(\lambda, u) = t(\lambda, u)t(\lambda, w)$ on H.

Let $W = V_{\Lambda_1}(z_1) \otimes \cdots \otimes V_{\Lambda_N}(z_N)$ be a tensor product of evaluation modules of $E_{\tau,\eta}(\widehat{sl}_2)$ and let $\Lambda = \Lambda_1 + \cdots + \Lambda_N$. Then, the highest weight state of the space H with $W[\Lambda] = Cv_0$ obeys the following highest weight condition,

$$\begin{aligned}
&c(\lambda, w)v_0 = 0, \quad a(\lambda, w)v_0 = v_0,\\
&d(\lambda, w) = \frac{\theta(\lambda - 2\eta\Lambda)}{\theta(\lambda)} \prod_{j=1}^{N} \frac{\theta(w - z_j - (-\Lambda + 1)\eta)}{\theta(w - z_j - (\Lambda + 1)\eta)} v_0.
\end{aligned} \tag{II.7}$$

Under the framework of the algebraic Bethe ansatz, the Bethe state is defined by

$$\Phi(t_1, \cdots, t_M) = b(t_1)b(t_2)\cdots b(t_M)v \equiv b(t_1)b(t_2)\cdots b(t_M)g(\lambda)v_0, \tag{II.8}$$

where $g(\lambda) \neq 0$ is a meromorphic function.

According to Felder and Varchenko [16, 18], W is a highest weight module of $E_{\tau,\eta}(\widehat{sl}_2)$ with $\Lambda = 2M \in 2Z_{\geq 0}$. Let $v(\lambda) = \prod_{j=1}^{M} \theta(\lambda - 2\eta j)$. Then applying the transfer matrix (II.6) to the Bethe state gives rise to

$$\begin{aligned}
& t(\lambda, w)\Phi(t_1, \cdots, t_M)\\
= \;& (a(\lambda, w) + d(\lambda, w))\Phi(t_1, \cdots, t_M)\\
= \;& \epsilon(\lambda, w)\Phi(t_1, \cdots, t_M) + \sum_{j=\alpha}^{M} xF_\alpha \Phi_\alpha(t_1, \cdots, t_{\alpha-1}, w, t_{\alpha+1}, t_M),
\end{aligned} \tag{II.9}$$

where F_α is a function of t_α, $x = \theta(\lambda + t_\alpha - w)\theta(2\eta)/(\theta(t_\alpha - w)\theta(\lambda - 2\eta))$ and

$$\Phi_\alpha(t_1, \cdots, t_{\alpha-1}, w, t_{\alpha+1}, t_M) = b(\lambda, t_1) \cdots b(\lambda, t_{\alpha-1}) b(\lambda, w) b(\lambda, t_{\alpha+1}) \cdots b(\lambda, t_M) v.$$

It follows that the Bethe state is an eigenstate of the transfer matrix $t(\lambda, w)$ if $F_\alpha = 0$. Therefore the eigenvalue of the transfer matrix is

$$\epsilon(\lambda, w) = \prod_{\alpha=1}^{M} \frac{\theta(t_\alpha - w - 2\eta)}{\theta(t_\alpha - w)} + \prod_{\alpha=1}^{M} \frac{\theta(t_\alpha - w + 2\eta)}{\theta(t_\alpha - w)} \prod_{k=1}^{N} \frac{\theta(w - z_k - (-\Lambda_k + 1)\eta)}{\theta(w - z_k - (\Lambda_k + 1)\eta)} \tag{II.10}$$

with t_α determined by the Bethe ansatz equations,

$$F_\alpha = 1 - \prod_{\beta \neq \alpha}^{M} \frac{\theta(t_\beta - t_\alpha - 2\eta)}{\theta(t_\beta - t_\alpha + 2\eta)} \prod_{k=1}^{N} \frac{\theta(t_\alpha - z_k - (-\Lambda_k + 1)\eta)}{\theta(t_\alpha - z_k - (\Lambda_k + 1)\eta)} = 0. \tag{II.11}$$

II.2 $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ and its Bethe ansatz

The boundary elliptic quantum group $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ is generated by h and the elements of the matrix $\mathcal{L}(\lambda, w) \in \mathrm{End}(C^2)$, $\mathcal{A}(\lambda, w)$, $\mathcal{B}(\lambda, w)$, $\mathcal{C}(\lambda, w)$ and $\mathcal{D}(\lambda, w)$ with non-commutative entries, subject to the relations [17]

$$\begin{aligned} &R_{21}(\lambda, w_1 - w_2)\mathcal{L}_1(\lambda - 2\eta h^{(2)}, w_1) R_{12}(\lambda, w_1 + w_2) \mathcal{L}_2(\lambda - 2\eta h^{(1)}, w_2) \\ &\quad = \mathcal{L}_2(\lambda - 2\eta h^{(1)}, w_2) R_{21}(\lambda, w_1 + w_2) \mathcal{L}_1(\lambda - 2\eta h^{(2)}, w_1) R_{12}(\lambda, w_1 - w_2). \end{aligned} \tag{II.12}$$

Let $V_\Lambda(z)$ be an evaluation module of $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ with the basis e_k, $k \in Z_{\geq 0}$. This module can be identified with the original representation space of $E_{\tau,\eta}(\widehat{sl}_2)$. On the module $V_\Lambda(z)$, the representation of $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ is given by

$$f(h) e_k = f(\Lambda - 2k) e_k, \tag{II.13}$$

$$\begin{aligned} \mathcal{A}(\lambda, w) e_k = {} & \{ Y(\lambda + 2\eta, -w) g(a, \lambda + 2\eta, w, k) g(d, \lambda + 2\eta, -w, k - 1) \\ & - K(\lambda - 2\eta(\Lambda - 2k + 2), w)_{22} Y(\lambda + 2\eta, -w) \\ & \times g(c, \lambda + 2\eta, w, k) g(a, \lambda + 2\eta, -w, k - 1) \} e_k, \end{aligned} \tag{II.14}$$

$$\begin{aligned} \mathcal{B}(\lambda, w) e_k = {} & \{ Y(\lambda + 2\eta, -w) g(b, \lambda + 2\eta, w, k) g(d, \lambda + 2\eta, -w, k) \\ & - K(\lambda - 2\eta(\Lambda - 2k), w)_{22} Y(\lambda + 2\eta, -w) \\ & \times g(d, \lambda + 2\eta, w, k) g(b, \lambda + 2\eta, -w, k) \} e_{k+1}, \end{aligned} \tag{II.15}$$

$$\begin{aligned} \mathcal{C}(\lambda, w) e_k = {} & \{ K(\lambda - 2\eta(\Lambda - 2k + 2), w)_{22} Y(\lambda - 2\eta, -w) X(\lambda - 2\eta, k - 1) \\ & \times g(c, \lambda - 2\eta, w, k) g(a, \lambda - 2\eta, -w, k) \\ & - Y(\lambda - 2\eta, -w) X(\lambda - 2\eta, k - 1) \\ & \times g(a, \lambda - 2\eta, w, k) g(c, \lambda - 2\eta, -w, k) \} e_{k-1}, \end{aligned} \tag{II.16}$$

$$
\begin{aligned}
\mathcal{D}(\lambda,w)e_k &= \{K(\lambda-2\eta(\Lambda-2k),w)_{22}Y(\lambda-2\eta,-w)X(\lambda-2\eta,k)\\
&\quad\times g(d,\lambda-2\eta,w,k)g(a,\lambda-2\eta,-w,k+1)\\
&\quad-Y(\lambda-2\eta,-w)X(\lambda-2\eta,k)\\
&\quad\times g(b,\lambda-2\eta,w,k)g(c,\lambda-2\eta,w,k+1)\}e_k,
\end{aligned}
\tag{II.17}
$$

where ξ is an arbitrary parameter, $g(i,\lambda,w,k),\ \ i=a,b,c,d$, are given by (II.5),

$$
\begin{aligned}
X(\lambda,k) &= \frac{\theta(\lambda-2(\Lambda+1-2k)\eta)}{\theta(\lambda-2(\Lambda-1-2k)\eta)},\\
Y(\lambda,w) &= \frac{\theta(z-w+(\Lambda+1)\eta)\theta(\lambda)}{\theta(z-w-(\Lambda+1)\eta)\theta(\lambda-2(\Lambda+1)\eta)},
\end{aligned}
$$

and $K(\lambda,w)_{22}$ is the (2,2) element of the K-matrix

$$
K(\lambda,w)=\mathrm{diag}\left(1,\frac{\theta(w+\xi)\theta(w+\lambda-\xi)}{\theta(w-\xi)\theta(w-\lambda+\xi)}\right)
\tag{II.18}
$$

which satisfies the boundary dynamical YBE:

$$
\begin{aligned}
&R_{21}(\lambda,w_1-w_2)K_1(\lambda-2\eta h^{(2)},w_1)R_{12}(\lambda,w_1+w_2)K_2(\lambda-2\eta h^{(1)},w_2)\\
&\quad=K_2(\lambda-2\eta h^{(1)},w_2)R_{21}(\lambda,w_1+w_2)K_1(\lambda-2\eta h^{(2)},w_1)R_{12}(\lambda,w_1-w_2).
\end{aligned}
\tag{II.19}
$$

Similar to the $E_{\tau,\eta}(\widehat{sl}_2)$ case, for any module $W=V_\Lambda$ of $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$, the boundary transfer matrix is defined by

$$
t^b(\lambda,w)=\mathrm{tr}_0K^+(\lambda,w)\mathcal{L}(\lambda,w),
\tag{II.20}
$$

where

$$
K^+(\lambda,w)=\mathrm{diag}\left(1,\frac{\theta(\lambda-2\eta)}{\theta(\lambda+2\eta)}\frac{\theta(w-2\eta-\xi)}{\theta(w-2\eta+\xi)}\frac{\theta(w-2\eta-\lambda+\xi)}{\theta(w-2\eta+\lambda-\xi)}\right)
\tag{II.21}
$$

is a diagonal solution to the dual boundary dynamical YBE,

$$
\begin{aligned}
&R_{21}(\lambda,-w_1+w_2)K_1^+(\lambda-2\eta h^{(2)},w_1)\tilde{R}_{12}(\lambda,-w_1-w_2+4\eta)K_2^+(\lambda-2\eta h^{(1)},w_2)\\
&\quad=K_2^+(\lambda-2\eta h^{(1)},w_2)\tilde{R}_{21}(\lambda,-w_1-w_2+4\eta)K_1^+(\lambda-2\eta h^{(2)},w_1)\\
&\qquad\times R_{12}(\lambda,-w_1+w_2)
\end{aligned}
\tag{II.22}
$$

with

$$
\begin{aligned}
\tilde{R}(\lambda,w) &= E_{0,0}\otimes E_{0,0}+E_{1,1}\otimes E_{1,1}+\alpha(\lambda,w)E_{0,0}\otimes E_{1,1}+\alpha(-\lambda,w)E_{1,1}\otimes E_{0,0}\\
&\quad+\frac{\theta(\lambda-2\eta)}{\theta(\lambda+2\eta)}\beta(\lambda,w)E_{0,1}\otimes E_{1,0}+\frac{\theta(\lambda+2\eta)}{\theta(\lambda-2\eta)}\beta(-\lambda,w)E_{1,0}\otimes E_{0,1}.
\end{aligned}
\tag{II.23}
$$

The transfer matrix (II.20) preserves the space $H=Fun(W)[0]$ of functions with values in the zero weight space $W[0]$. Let $W=V_{\Lambda_1}(z_1)\otimes\cdots V_{\Lambda_N}(z_N)$ be a tensor

product of evaluation modules of $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$, and let $\Lambda = \Lambda_1 + \cdots + \Lambda_N$. The highest weight vector v_0 of H with $W[\Lambda] = Cv_0$ satisfies the highest weight condition:

$$\begin{aligned}
\mathcal{C}(\lambda,w)v_0 &= 0, \qquad \mathcal{A}(\lambda,w)v_0 = v_0,\\
\tilde{\mathcal{D}}(\lambda,w)v_0 &= \frac{\theta(2w)\theta(w-\xi+\lambda-2(\Lambda+1)\eta)\theta(w+\xi-2\eta)}{\theta(2w-2\eta)\theta(w+\xi-(\lambda-2\Lambda\eta))\theta(w-\xi)}\\
&\quad\times\prod_{j=1}^{N}\frac{\theta(z_j+w+(\Lambda_j-1)\eta)\theta(z_j-w-(\Lambda_j-1)\eta)}{\theta(z_j+w-(\Lambda_j+1)\eta)\theta(z_j-w+(\Lambda_j+1)\eta)}v_0, \qquad \text{(II.24)}
\end{aligned}$$

where $\tilde{\mathcal{D}}(\lambda,w)$ is determined from the relation: $\mathcal{D}(\lambda,w) = \theta(\lambda)/\theta(\lambda-2\eta)\tilde{\mathcal{D}}(\lambda,w) + \beta(\lambda-2\eta,2w)\mathcal{A}(\lambda,w)$. The space $H = Fun(W)$ is spanned by the Bethe state

$$\Phi^b(t_1,\cdots,t_M) = \mathcal{B}(t_1)\cdots\mathcal{B}(t_M)v_0. \qquad \text{(II.25)}$$

Applying the boundary transfer matrix to the Bethe state, one obtains

$$t^b(\lambda,w)\Phi^b(t_1,\cdots,t_M) = \epsilon(\lambda,w)\Phi^b(t_1,\cdots,t_M) + \sum_{\alpha=1}^{M} xF^b_\alpha\Phi^b_\alpha(t_1,\cdots,t_{\alpha-1},w,t_{\alpha+1},t_M), \qquad \text{(II.26)}$$

where

$$x = \frac{\theta(2\eta)\theta(2t_\alpha)}{\theta(\lambda+2\eta)\theta(2t_\alpha-2\eta)}\left[\frac{\theta(2w-4\eta)\theta(w+t_\alpha-\lambda)}{\theta(2w-2\eta)\theta(w+t_\alpha-2\eta)} - \frac{\theta(w-t_\alpha-\lambda-2\eta)}{\theta(w-t_\alpha)}\right],$$

$$\begin{aligned}
\epsilon(\lambda,w) &= \prod_{\alpha=1}^{M}\frac{\theta(w+t_\alpha)\theta(w-t_\alpha+2\eta)}{\theta(w+t_\alpha-2\eta)\theta(w-t_\alpha)} + \frac{\theta(w-2\eta-\xi)\theta(w-\lambda-2\eta+\xi)}{\theta(w+\lambda-\xi)\theta(w+\xi)}\\
&\quad\times\frac{\theta(2w)\theta(w-2\eta+\xi)\theta(w+\lambda-2(\Lambda+1)\eta-\xi)}{\theta(2w-4\eta)\theta(w-\xi)\theta(w-\lambda+2\Lambda\eta+\xi)}\\
&\quad\times\prod_{\alpha=1}^{M}\frac{\theta(w+t_\alpha-4\eta)\theta(w-t_\alpha-2\eta)}{\theta(w+t_\alpha-2\eta)\theta(w-t_\alpha)}\\
&\quad\times\prod_{l=1}^{N}\frac{\theta(z_l+w+(\Lambda_l-1)\eta)\theta(z_l-w-(\Lambda_l-1)\eta)}{\theta(z_l+w-(\Lambda_l+1)\eta)\theta(z_l-w+(\Lambda_l+1)\eta)}, \qquad \text{(II.27)}
\end{aligned}$$

and F^b_α obeys the Bethe ansatz equations:

$$\begin{aligned}
F^b_\alpha &= 1 - \prod_{\beta=1,\neq\alpha}^{M}\frac{\theta(t_\alpha-t_\beta-2\eta)\theta(t_\alpha+t_\beta-4\eta)}{\theta(t_\alpha+t_\beta)\theta(t_\alpha-t_\beta+2\eta)}\\
&\quad\times\frac{\theta(t_\alpha-2\eta-\xi)\theta(t_\alpha-2\eta-\lambda+\xi)}{\theta(t_\alpha+\lambda-\xi)\theta(t_\alpha+\xi)}\\
&\quad\times\frac{\theta(t_\alpha-2\eta+\xi)\theta(t_\alpha-2(\Lambda+1)\eta+\lambda-\xi)}{\theta(t_\alpha-\lambda+2\Lambda\eta+\xi)\theta(t_\alpha-\xi)}\\
&\quad\times\prod_{l=1}^{N}\frac{\theta(z_l+t_\alpha+(\Lambda_l-1)\eta)\theta(z_l-t_\alpha-(\Lambda_l-1)\eta)}{\theta(z_l+t_\alpha-(\Lambda_l+1)\eta)\theta(z_l-t_\alpha+(\Lambda_l+1)\eta)}\\
&= 0. \qquad \text{(II.28)}
\end{aligned}$$

III Gaudin magnets and KZ equations based on the elliptic quantum groups

III.1 Periodic Gaudin magnet

The Gaudin model was proposed as a quantum integrable system with long-range interactions [1]. Such a system can be solved by using the algebraic Bethe ansatz method. By taking the quasi-classical limit $\eta \to 0$ of the transfer matrix of the six-vertex model, Hikami gave the Hamiltonian of the XXZ Gaudin model [13].

Foe our case, we expand the transfer matrix (II.20) around the point $\eta = 0$ with the parameter $w = z_j$, to obtain

$$t(\lambda, w = z_j) = \frac{2}{\Lambda_j + 1}(1 + \eta H_j + O(\eta^2)). \tag{III.1}$$

One can prove $[H_j, H_k]$=0 since the first term on the right hand side of (III.1) is a constant. Therefore, H_j gives the Hamiltonian associated with $E_{\tau,\eta}(\widehat{sl}_2)$. From (II.6), we find

$$\begin{aligned} H_j &= \left.\frac{dt(\lambda, z_j)}{d\eta}\right|_{\eta=0} \\ &= \sum_{l=1,\neq j}^{N} \Big[W_1(z_l, z_j)E_l^- E_j^+ + W_2(z_l, z_j)E_l^+ E_j^- - 2h_j\partial_\lambda \\ &\quad + \frac{1}{2}\varphi(\lambda)(\Lambda_l h_j + h_l) + \frac{1}{2}\varphi(z_j - z_l)(h_l h_j - \Lambda_l)\Big], \end{aligned} \tag{III.2}$$

where $\varphi(x) \equiv \theta'(x)/\theta(x)$,

$$\begin{aligned} W_1(x, y) &= -2\frac{\theta(-\lambda + x - y)\theta'(0)}{\theta(x - y)\theta(\lambda)}, \\ W_2(x, y) &= -2\frac{\theta(-\lambda - x + y)\theta'(0)}{\theta(x - y)\theta(\lambda)}, \end{aligned} \tag{III.3}$$

and h, $E^\pm$ are defined by

$$\begin{aligned} &h_l e_k = (\Lambda_l - 2k)e_k, \\ &E_l^- e_k = e_{k+1}, \quad E_l^+ e_k = k(\Lambda_l + 1 - k)e_{k-1}. \end{aligned} \tag{III.4}$$

We can check that h, $E^\pm$ satisfy the commutation relations

$$[h_i, E_j^\pm] = \pm 2\delta_{i,j}E_j^\pm, \quad [E_i^+, E_j^-] = \delta_{i,j}h_j. \tag{III.5}$$

The eigenvalues ϵ_j of H_j can be extracted from

$$\epsilon(\lambda, w = z_j) = \frac{2}{\Lambda_j + 1}(1 + \eta\epsilon_j + O(\eta^2)), \tag{III.6}$$

giving

$$\epsilon_j = -2\Lambda_j \sum_{\alpha=1}^{M} \varphi(t_\alpha - z_j) + (1-\Lambda_j) \sum_{l=1,\neq j}^{N} \Lambda_l \varphi(z_j - z_l). \tag{III.7}$$

And the constraints for the eigenvalues are $f_\alpha = 0,\ \alpha = 1, 2, \cdots, M$, with

$$f_\alpha = \sum_{\beta\neq\alpha}^{M} 4\varphi(t_\beta - t_\alpha) + 2\sum_{l=1}^{N} \Lambda_l \varphi(t_\alpha - z_l). \tag{III.8}$$

We now define the following equivalent Hamiltonian $\mathcal{H}_j$ of the Gaudin model by shifting H_j by some constants,

$$\mathcal{H}_j = H_j - (1-\Lambda_j) \sum_{l=1,\neq j}^{N} \Lambda_l \varphi(z_j - z_l). \tag{III.9}$$

The corresponding eigenvalues are

$$\mathcal{E}_j = -2\Lambda_j \sum_{\alpha=1}^{M} \varphi(t_\alpha - z_j). \tag{III.10}$$

Then, from (II.9), we obtain the so-called off-shell Bethe ansatz equations:

$$\mathcal{H}_j \phi = \mathcal{E}_j \phi + \sum_{\alpha=1}^{M} f_\alpha W_1(z_j, t_\alpha) E_j^- \phi_\alpha, \tag{III.11}$$

where ϕ, ϕ_α are Bethe states for the Gaudin model given by

$$\begin{aligned} \phi &\equiv \prod_{\alpha=1}^{M} \left.\frac{db(\lambda, t_\alpha)}{d\eta}\right|_{\eta=0} |0>, \\ \phi_\alpha &\equiv \prod_{\beta=1,\neq\alpha}^{M} \left.\frac{db(\lambda, t_\beta)}{d\eta}\right|_{\eta=0} |0> . \end{aligned} \tag{III.12}$$

By using the representation (II.4) of $E_{\tau,\eta}(\widehat{sl}_2)$, we derive the precise form of the Bethe states

$$\begin{aligned} \phi &= \prod_{\alpha=1}^{M} \sum_{k=1}^{N} W_1(z_k, t_\alpha) E_k^- |0>, \\ \phi_\alpha &= \prod_{\beta\neq\alpha}^{M} \sum_{k=1}^{N} W_1(z_k, t_\beta) E_k^- |0> . \end{aligned} \tag{III.13}$$

III.2 Elliptic KZ equation

As integrable differential equations, the KZ equations take the form

$$\nabla_j \Psi = 0 \qquad \text{for} \quad j = 1, 2, \cdots, N, \tag{III.14}$$

where the differential operators ∇_j are defined by the Gaudin model Hamiltonian $\mathcal{H}_j$

$$\nabla_j = \kappa \frac{\partial}{\partial z_j} - \mathcal{H}_j \tag{III.15}$$

with κ being an arbitrary parameter. Substituting (III.2) into (III.15), we can check

$$[\nabla_j, \nabla_k] = 0, \tag{III.16}$$

which ensures the integrability of the KZ equations.

The function $\Psi(z)$ can be constructed by the hypergeometric function $\chi(z,t)$ which is a solution of the following equations

$$\begin{aligned} \kappa \frac{\partial}{\partial z_j} \chi &= \mathcal{E}_j \chi, \\ \kappa \frac{\partial}{\partial t_\alpha} \chi &= f_\alpha \chi. \end{aligned} \tag{III.17}$$

The hypergeometric function $\chi(z,t)$ can be written as

$$\chi(z,t) = \prod_{\beta<\alpha} [\theta(t_\alpha - t_\beta)]^{-4/\kappa} \prod_{\alpha=1}^{M} \prod_{j=1}^{N} [\theta(z_j - t_\alpha)]^{2\Lambda_j/\kappa}. \tag{III.18}$$

With the help of $\chi(z,t)$, we obtain

$$\Psi(z) = \oint_C \cdots \oint_C dt_1 \cdots dt_M \chi(t,z) \phi(t,z), \tag{III.19}$$

where the integration path C is a closed contour in the Riemann surface such that the integrand resumes its initial value.

Substituting the expressions of ∇_j and $\Psi(z)$ into (III.14), we can show that the KZ equations are satisfied. The proof is as follows

$$\begin{aligned} \kappa \frac{\partial}{\partial z_j} \Psi(z) &= \oint_C \cdots \oint_C dt_1 \cdots dt_M \left(\kappa \frac{\partial \chi}{\partial z_j} \phi + \kappa \chi \frac{\partial \phi}{\partial z_j} \right) \\ &= \oint_C \cdots \oint_C dt_1 \cdots dt_M \left(\chi \mathcal{E}_j \phi + \kappa \chi \frac{\partial \phi}{\partial z_j} \right) \\ &= \oint_C \cdots \oint_C dt_1 \cdots dt_M \left[\chi \mathcal{H}_j \phi - \kappa \sum_{\alpha=1}^{M} \frac{\partial \chi}{\partial t_\alpha} W_1(z_j, t_\alpha) E_j^- \phi_\alpha \right. \\ &\quad \left. - \kappa \chi \sum_{\alpha=1}^{M} \frac{\partial}{\partial t_\alpha} \left(W_1(z_j, t_\alpha) E_j^- \phi_\alpha \right) \right] \\ &= \oint_C \cdots \oint_C dt_1 \cdots dt_M \left[\mathcal{H}_j \chi \phi - \kappa \sum_{\alpha=1}^{M} \frac{\partial}{\partial t_\alpha} \left(\chi W_1(z_j, t_\alpha) E_j^- \phi_\alpha \right) \right] \\ &= \mathcal{H}_j \Psi(z), \end{aligned} \tag{III.20}$$

III.3 Boundary Gaudin magnet and KZ equation

The Gaudin model Hamiltonian for $\mathcal{B}E_{\tau,\eta}(\widehat{sl}_2)$ is found to be the form [19]

$$\begin{aligned} H_j^b(\lambda, z_j) &= \delta(\lambda, z_j, \xi) \\ &+ \sum_{k=1}^{N} \Big[K_{22} W_1^+(z_k, z_j) E_k^- E_j^+ + K_{22}^+ W_2^+(z_k, z_j) E_k^+ E_j^- \\ &\quad - \varphi(z_k + z_j)(h_k h_j - \Lambda_k) + \varphi(\lambda)(2\Lambda_k - h_k - \Lambda_k h_j)] \\ &+ \sum_{k=1,\neq j}^{n} \Big[W_1^-(z_k, z_j) E_k^- E_j^+ + W_2^-(z_k, z_j) E_k^+ E_j^- \\ &\quad + \varphi(z_k - z_j)(h_k h_j - \Lambda_k) - \varphi(\lambda)(h_k - \Lambda_k h_j)] \,, \end{aligned} \tag{III.21}$$

where $E^{\pm}, h$ and φ are the same as in the periodic case, K_{22} and K_{22}^+ are $(2,2)$-elements of K and K^+ respectively, and

$$\begin{aligned} W_1^{\pm}(x, y) &= -2\frac{\theta(-\lambda + x \pm y)\theta'(0)}{\theta(x \pm y)\theta(\lambda)}, \\ W_2^{\pm}(x, y) &= 2\frac{\theta(\lambda + x \pm y)\theta'(0)}{\theta(x \pm y)\theta(\lambda)}, \\ \delta(\lambda, z_j, \xi) &= (1 + 2h_j)(h_j - 1)[\varphi(z_j - \lambda + \xi) - \varphi(z_j + \lambda - \xi)] \\ &\quad + (h_j - 1)[2\varphi(\lambda) + \varphi(z_j - \xi) - \varphi(z_j + \xi)] \\ &\quad + h_j(\Lambda_j - 1)\left[\varphi(\lambda) + \frac{\varphi''(0)}{\varphi'(0)}\right], \end{aligned} \tag{III.22}$$

By (II.27) we find the eigenvalues of the boundary Gaudin model Hamiltonian H_j^b,

$$\begin{aligned} \epsilon_j^b &= \left.\frac{d\epsilon(\lambda, w = z_j)}{d\eta}\right|_{\eta=0} \\ &= \sum_{\alpha=1}^{M} 2\Lambda_j[\varphi(z_j + t_\alpha) + \varphi(z_j - t_\alpha)] - (1 - \Lambda_j^2)\varphi(2z_j) \\ &\quad + \sum_{l=1,\neq j}^{N} \Lambda_l(1 - \Lambda_j)[\varphi(z_l + z_j) - \varphi(z_l - z_j)] \\ &\quad - (1 - \Lambda_j)[\varphi(z_j + \xi) + \varphi(z_j - \xi) + \varphi(z_j - \lambda + \xi) + \varphi(z_j + \lambda - \xi)] \end{aligned} \tag{III.23}$$

with the constraints $f_\alpha^b = 0, \ \alpha = 1, 2, \cdots, M$, where

$$\begin{aligned} f_\alpha^b &= 2\varphi(t_\alpha - \xi) + 2\varphi(t_\alpha + \xi) + 2(1 + \Lambda)[\varphi(t_\alpha - \lambda + \xi) + \varphi(t_\alpha + \lambda - \xi)] \\ &\quad + 4\sum_{\beta=1,\neq\alpha}^{M}[\varphi(t_\alpha - t_\beta) + \varphi(t_\alpha + t_\beta)] - \sum_{l=1}^{N} 2\Lambda_l[\varphi(z_l + t_\alpha) - \varphi(z_l - t_\alpha)]. \end{aligned} \tag{III.24}$$

The Bethe states for the boundary Gaudin model are then given by

$$\begin{aligned}
\phi^b &= \prod_{\alpha=1}^{M}\left[\sum_{l=1}^{N}(W_1^-(z_l,t_\alpha)-K_{22}(t_\alpha)W_1^+(z_l,t_\alpha))E_l^-\right]v_0,\\
\phi^b_\alpha &= \prod_{\beta=1,\neq\alpha}^{M}\left[\sum_{l=1}^{N}(W_1^-(z_l,t_\beta)-K_{22}(t_\beta)W_1^+(z_l,t_\beta))E_l^-\right]v_0. \qquad \text{(III.25)}
\end{aligned}$$

As in the previous subsection, we define the following equivalent Hamiltonian $\mathcal{H}_j^b$ of the boundary Gaudin model:

$$\begin{aligned}
\mathcal{H}_j^b &= -H_j^b+\sum_{l=1,\neq j}^{N}2(\Lambda_l)(1-\Lambda_j)[\varphi(z_l+z_j)-\varphi(z_l-z_j)]\\
&\quad-(1-\Lambda_j)[z_j+\xi)+\varphi(z_j-\xi)+\varphi(z_j-\lambda+\xi)+\varphi(z_j+\lambda-\xi)]. \qquad \text{(III.26)}
\end{aligned}$$

The corresponding eigenvalues are given by

$$\mathcal{E}_j^b=(1-\Lambda_j^2)\varphi(2z_j)-\sum_{\alpha=1}^{M}2\Lambda_j[\varphi(z_j+t_\alpha)+\varphi(z_j-t_\alpha)], \qquad \text{(III.27)}$$

and the off-shell Bethe ansatz equations read

$$\mathcal{H}_j^b\phi^b=\mathcal{E}_j^b\phi^b+\sum_{\alpha=1}^{M}f_\alpha^b\left(W_1^-(z_l,t_\alpha)-W_1^+(z_l,t_\alpha)\right)E_j^-\phi_\alpha^b. \qquad \text{(III.28)}$$

Thus the KZ equations based on the boundary Gaudin model are given by

$$\nabla_j\Psi^b=0 \quad \text{for } j=1,2,\cdots,N, \qquad \text{(III.29)}$$

where the differential operators ∇_j are defined by

$$\nabla_j=\kappa\frac{\partial}{\partial z_j}-\mathcal{H}_j^b \qquad \text{(III.30)}$$

with κ being an arbitrary parameter. To ensure the integrability of the KZ equation, we impose

$$[\nabla_j,\nabla_k]=0, \qquad \text{(III.31)}$$

This requires $\xi\to\infty$. With this condition, the Hamiltonian and the Bethe states of the boundary Gaudin model become

$$\begin{aligned}
\mathcal{H}_j^b(\lambda) &= -\Delta(\lambda,z_j,\xi)-\sum_{k=1}^{N}\Big[W_1^+E_k^-E_j^++W_2^+E_k^+E_j^-\\
&\quad-\ \varphi(z_k+z_j)(h_kh_j-\Lambda_k)+\varphi(\lambda)(2\Lambda_k-h_k-\Lambda_kh_j)]\\
&\quad-\sum_{k=1,\neq j}^{n}\Big[W_1^-E_k^-E_j^++W_2^-E_k^+E_j^-
\end{aligned}$$

$$
\begin{aligned}
&\quad + \ \varphi(z_k - z_j)(h_k h_j - \Lambda_k) - \varphi(\lambda)(h_k - \Lambda_k h_j)\right], \\
\phi^b &= \prod_{\alpha=1}^{M}\left[\sum_{l=1}^{N}(W_1^-(z_l,t_\alpha) - W_1^+(z_l,t_\alpha))E_l^-\right] v_0, \\
\phi^b_\alpha &= \prod_{\beta=1,\neq\alpha}^{M}\left[\sum_{l=1}^{N}(W_1^-(z_l,t_\beta) - W_1^+(z_l,t_\beta))E_l^-\right] v_0,
\end{aligned} \tag{III.32}
$$

where

$$
\begin{aligned}
\Delta(\lambda) &= \sum_{l=1,\neq j}^{N} 2(\Lambda_l)(1-\Lambda_j)[\varphi(z_l+z_j) - \varphi(z_l - z_j)] \\
&\quad + [h_j(\Lambda_j - 1)]\varphi(\lambda) + (\Lambda_j + h_j)(\Lambda_j - 1)\frac{\varphi''(0)}{\varphi'(0)}.
\end{aligned}
$$

The function Ψ^b can be computed by using the following hypergeometric type integral

$$
\Psi^b(z) = \oint_C \cdots \oint_C dt_1 \cdots dt_M \chi^b(t,z)\phi^b(t,z), \tag{III.33}
$$

where the integration path C is a closed contour in the Riemann surface, similar to that described in the previous subsection. The hypergeometric kernel $\chi^b(t,z)$ is given by

$$
\begin{aligned}
\chi^b(t,z) &= \prod_{j=1}^{N}(\theta(2z_j))^{(1-\Lambda_j^2)/2\kappa}\prod_{j=1}^{N}\prod_{\alpha=1}^{M}[\theta(z_j - t_\alpha)\theta(z_j + t_\alpha)]^{-2\Lambda_j/\kappa} \\
&\quad \times \prod_{\alpha<\beta}^{M}[\theta(t_\alpha - t_\beta)\theta(t_\alpha + t_\beta)]^{4/\kappa}.
\end{aligned} \tag{III.34}
$$

One can check that the kernel $\chi^b(t,z)$ satisfies the following equations

$$
\begin{aligned}
\kappa\frac{\partial}{\partial z_j}\chi^b &= \mathcal{E}_j^b\chi, \\
\kappa\frac{\partial}{\partial t_\alpha}\chi^b &= f_\alpha^b\chi.
\end{aligned} \tag{III.35}
$$

Moreover we can show that Ψ^b satisfies the KZ equations,

$$
\nabla_j\Psi^b(z) = 0. \tag{III.36}
$$

The proof is as follows.

$$
\begin{aligned}
\kappa\frac{\partial}{\partial z_j}\Psi(z) &= \oint_C \cdots \oint_C dt_1 \cdots dt_M\left(\kappa\frac{\partial\chi^b}{\partial z_j}\phi^b + \kappa\chi^b\frac{\partial\phi^b}{\partial z_j}\right) \\
&= \oint_C \cdots \oint_C dt_1 \cdots dt_M\left(\chi\mathcal{E}_j^b\phi^b + \kappa\chi^b\frac{\partial\phi^b}{\partial z_j}\right) \\
&= \oint_C \cdots \oint_C dt_1 \cdots dt_M\left[\chi^b\mathcal{H}_j^b\phi^b\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\kappa \sum_{\alpha=1}^{M} \frac{\partial \chi^b}{\partial t_\alpha} \left(W_1^-(z_j,t_\alpha) - W_1^+(z_j,t_\alpha)\right) E_j^- \phi_\alpha^b \\
& -\kappa \chi^b \sum_{\alpha=1}^{M} \frac{\partial}{\partial t_\alpha} \left(\left(W_1^-(z_j,t_\alpha) - W_1^+(z_j,t_\alpha)\right) E_j^- \phi_\alpha^b\right)\Big] \\
= & \oint_C \cdots \oint_C dt_1 \cdots dt_M \Big[\mathcal{H}_j^b \chi\phi \\
& -\kappa \sum_{\alpha=1}^{M} \frac{\partial}{\partial t_\alpha} \left(\chi^b \left(W_1^-(z_j,t_\alpha) - W_1^+(z_j,t_\alpha)\right) E_j^- \phi_\alpha^b\right)\Big] \\
= & \mathcal{H}_j^b \Psi(z), \qquad \text{(III.37)}
\end{aligned}
$$

IV XYZ Gaudin magnets

The XYZ chain or eight-vertex model occupies an important place in the study of integrable systems [20]. The periodic XYZ Gaudin model has been constructed and diagonalized in [5] by means of the algebraic Bethe ansatz method. The corresponding KZ equations and their solutions were given in [21]. Here we write down a more explicit formula for the Bethe states for the periodic case. Moreover, we present the Hamiltonian, eigenvalues and (off-shell) Bethe ansatz equations for the XYZ Gaudin magnet with an integrable boundary.

The Boltzmann weights of the eight-vertex model are given by

$$
\begin{aligned}
R(w) = & E_{00} \otimes E_{00} + E_{11} \otimes E_{11} + \kappa \mathrm{sn}(\eta)\mathrm{sn}(w)(E_{01} \otimes E_{01} + E_{10} \otimes E_{10}) \\
& + \frac{\mathrm{sn}(w)}{\mathrm{sn}(w+\eta)}(E_{00} \otimes E_{11} + E_{11} \otimes E_{00}) + \frac{\mathrm{sn}(\eta)}{\mathrm{sn}(w+\eta)}(E_{01} \otimes E_{10} + E_{10} \otimes E_{01}),
\end{aligned}
\tag{IV.1}
$$

where $\mathrm{sn}(w) \equiv \mathrm{sn}(w,\kappa)$ is the Jacobi elliptic function of modulus κ, $0 \le \kappa \le 1$. Define the corresponding transfer matrix as

$$
t(w) = tr_0 R_{01}(w - z_1) \cdots R_{0N}(w - z_N) \tag{IV.2}
$$

for the periodic case, and as

$$
\begin{aligned}
t^b(w) = & tr_0 K^+(w,\xi) R_{01}(w - z_1) \cdots R_{0N}(w - z_N) K(w,\xi) \\
& \times R_{0N}(w + z_N) \cdots R_{01}(w + z_1)
\end{aligned}
\tag{IV.3}
$$

for the boundary case, where $K(u,\xi) = \mathrm{diag}(\mathrm{sn}(\xi+u), \mathrm{sn}(\xi-u))/\mathrm{sn}(\xi)$ [22] and $K^+(u,\xi) = K(-u-\eta,\xi)$ with ξ being a parameter.

Then, by using the same method as in the previous sections, one can obtain the Hamiltonians of the XYZ Gaudin models.

(i) The periodic case:

$$
\begin{aligned}
H_j = & \sum_{k=1,\neq j}^{N} \frac{1}{\mathrm{sn}(z_j - z_k)} \Big[(1 + \kappa \mathrm{sn}^2(z_j - z_k)\sigma_j^x \sigma_k^x + (1 - \kappa \mathrm{sn}^2(z_j - z_k)\sigma_j^y \sigma_k^y \\
& + \mathrm{cn}(z_j - z_k)\mathrm{dn}(z_j - z_k)(\sigma_j^z \sigma_k^z - 1)\Big],
\end{aligned}
\tag{IV.4}
$$

where and throughout this paper, $\mathrm{cn}(u)\mathrm{dn}(u) = d\mathrm{sn}(u)/du$.

(ii) The open boundary case:

$$
\begin{aligned}
H_j^b \;=\;& -\frac{1}{2\mathrm{sn}(\xi - z_j)}\left[\mathrm{cn}(\xi - z_j)\mathrm{dn}(\xi - z_j) - \frac{\mathrm{sn}(\xi + z_j)}{\mathrm{sn}(2z_j)}\right](\sigma_j^z + 1) \\
& - \frac{1}{2\mathrm{sn}(\xi + z_j)}\left[\mathrm{cn}(\xi + z_j)\mathrm{dn}(\xi + z_j) + \frac{\mathrm{sn}(\xi - z_j)}{\mathrm{sn}(2z_j)}\right](\sigma_j^z - 1) \\
& + \sum_{k=1\neq j}^{N} \frac{1}{\mathrm{sn}(z_j - z_k)}\Big[\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+ + \kappa\mathrm{sn}^2(z_j - z_k)(\sigma_j^+\sigma_k^+ + \sigma_j^-\sigma_k^-) \\
& \quad + \mathrm{cn}(z_j - z_k)\mathrm{dn}(z_j - z_k)\frac{\sigma_j^z\sigma_k^z - 1}{2}\Big] \\
& + \sum_{k=1\neq j}^{N} \frac{1}{\mathrm{sn}(z_j + z_k)}\Big[\kappa\mathrm{sn}^2(z_j + z_k)\left(\frac{\mathrm{sn}(\xi + z_j)}{\mathrm{sn}(\xi - z_j)}\sigma_j^+\sigma_k^+ + \frac{\mathrm{sn}(\xi - z_j)}{\mathrm{sn}(\xi + z_j)}\sigma_j^-\sigma_k^-\right) \\
& \quad + \frac{\mathrm{sn}(\xi + z_j)}{\mathrm{sn}(\xi - z_j)}\sigma_j^+\sigma_k^- + \frac{\mathrm{sn}(\xi - z_j)}{\mathrm{sn}(\xi + z_j)}\sigma_j^-\sigma_k^+ + \mathrm{cn}(z_j + z_k)\mathrm{dn}(z_j + z_k)\frac{\sigma_j^z\sigma_k^z - 1}{2}\Big].
\end{aligned}
\tag{IV.5}
$$

The Hamiltonians (IV.4) and (IV.5) can be diagonalized by using the method similar to the previous section. The results are as follows.

(i) The periodic case.

The eigenvalues of (IV.4) are found to be

$$E_j = -2\sum_{\alpha=1}^{M}[\varphi_4(z_j - t_\alpha) + \varphi_1(z_j - t_\alpha)] \tag{IV.6}$$

with the constraints $f_\alpha = 0,\ \alpha = 1, 2, \cdots, M$, where

$$f_\alpha = -4\sum_{\beta=1,\neq\alpha}^{M}[\varphi_4(t_\beta - t_\alpha) + \varphi_1(t_\beta - t_\alpha)] - 2\sum_{j=1}^{N}[\varphi_4(t_\alpha - z_j) + \varphi_1(t_\alpha - z_j)]. \tag{IV.7}$$

Here $\varphi_4(z) \equiv \Theta'(z)/\Theta(z)$ and $\varphi_1(z) \equiv H'(z)/H(z)$. $\Theta(z)$ and $H(z)$ are the Jacobi theta-functions and they satisfy the relations

$$\mathrm{sn}(z) = H(z)/(\sqrt{k}\Theta(z)), \quad \mathrm{sn}^2(z) + \mathrm{cn}^2(z) = 1, \quad k^2\mathrm{sn}^2(z) + \mathrm{dn}^2(z) = 1.$$

We can write down the off-shell Bethe ansatz equations

$$H_j\phi = E_j\phi + \sum_{\alpha=1}^{M} x f_\alpha \frac{1}{\Theta(t_\alpha - z_j)H(t_\alpha - z_j)}N(t)\phi_\alpha, \tag{IV.8}$$

where

$$x = \frac{2g'(0)g(\frac{s+t}{2} + t_\alpha - z_j - \frac{1}{2})}{g(t_\alpha - z_j)g(\frac{s+t}{2} - \frac{1}{2})}$$

with s, t being parameters and $g(u) = H(u)\Theta(u)$,

$$N(t) = c(s,t) \cdot \begin{pmatrix} \Theta(t)H(t) & -H(t)H(t) \\ \Theta(t)\Theta(t) & -\Theta(t)H(t) \end{pmatrix}.$$

The Bethe states ϕ and ϕ_α are given by

$$\begin{aligned} \phi &= \prod_{\alpha=1}^{M}\sum_{j=1}^{N} \frac{1}{\Theta(t_\alpha - z_j)H(t_\alpha - z_j)} M(t, t_\alpha, z_j)|0>, \\ \phi_\alpha &= \prod_{\beta\neq\alpha}^{M}\sum_{j=1}^{N} \frac{1}{\Theta(t_\beta - z_j)H(t_\beta - z_j)} M(t, t_\beta, z_j)|0>, \end{aligned} \qquad \text{(IV.9)}$$

where

$$M(t, t_\beta, z_j) = \rho(s, t, t_\alpha, z_j) \cdot \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with

$$\begin{aligned} M_{11} = {} & 2\alpha\Theta(t_\alpha - z_j)H(t + t_\alpha - z_j)[\Theta'(t + t_\alpha - z_j)H(t + t_\alpha - z_j) \\ & - \Theta(t + t_\alpha - z_j)H'(t + t_\alpha - z_j)] \\ & + \Theta(t + t_\alpha - z_j)H(t + t_\alpha - z_j)[\Theta'(t_\alpha - z_j)H(t_\alpha - z_j) \\ & - \Theta(t_\alpha - z_j)H'(t_\alpha - z_j)], \\ M_{12} = {} & \frac{H'(0)}{\Theta(0)}[\Theta^2(t + t_\alpha - z_j)H^2(t_\alpha - z_j) - \Theta^2(t_\alpha - z_j)H^2(t + t_\alpha - z_j)], \\ M_{21} = {} & \frac{H'(0)}{\Theta(0)}[\Theta^2(t + t_\alpha - z_j)\Theta^2(t_\alpha - z_j) - H^2(t + t_\alpha - z_j)H^2(t_\alpha - z_j)], \\ M_{22} = {} & 2\alpha\Theta(t_\alpha - z_j)H(t + t_\alpha - z_j)[\Theta'(t + t_\alpha - z_j)H(t + t_\alpha - z_j) \\ & - \Theta(t + t_\alpha - z_j)H'(t + t_\alpha - z_j)] \\ & - \Theta(t + t_\alpha - z_j)H(t + t_\alpha - z_j)[\Theta'(t_\alpha - z_j)H(t_\alpha - z_j) \\ & - \Theta(t_\alpha - z_j)H'(t_\alpha - z_j)]. \end{aligned}$$

(ii) The boundary case.

For the boundary Gaudin model (IV.5) we find that the eigenvalues are given by

$$E_j^b = 2\varphi(2z_j) + a\varphi(z_j + \xi) - \sum_{\alpha=1}^{M}[\varphi(z_j - t_\alpha) + \varphi(z_j + t_\alpha) \qquad \text{(IV.10)}$$

with the constraints $f_\alpha^b = 0, \ \alpha = 1, 2, \cdots, M$, where

$$\begin{aligned} f_\alpha^b = {} & c[\varphi(t_\alpha - \xi) + \varphi(t_\alpha + \xi)] + 2\varphi(2t_\alpha) \\ & + 2\sum_{\beta\neq\alpha}^{M}[\varphi(t_\alpha - t_\beta) + \varphi(t_\alpha + t_\beta)] - \sum_{k=1}^{N}[\varphi(t_\alpha - z_k) + \varphi(t_\alpha + z_k)]. \end{aligned} \qquad \text{(IV.11)}$$

where a, c are some complex parameters. As previous section, the off-shell Bethe ansatz equations can be obtained by taking $\eta \to 0$ to the

$$t(z_j)\Phi^b = \mathcal{E}(z_j)\Phi^b + \sum_{\alpha=1}^{M} F^b_\alpha \Phi^b_\alpha. \qquad \text{(IV.12)}$$

Here $t(z_j), \mathcal{E}(z_j)$ and F^b_α can be found in [23], Φ^b, Φ^b_α are Bethe states given by

$$\begin{aligned} \Phi^b &= B(m^0-2|t_1)\cdots B(m^0-2M|z_M)|0>, \\ \Phi^b_\alpha &= B(m^0-2|z_j)B(m^0-4|t_1)\cdots B(m^0-2\alpha|t_{\alpha-1}) \\ &\quad \times B(m^0-2(\alpha-1)|t_{\alpha+1})\cdots B(m^0-2M|t_M)|0>, \end{aligned} \qquad \text{(IV.13)}$$

where m^0 is a parameter and B is the $(1,2)$ element of the 2×2 monodromy matrix which can be found in [23].

V Summary

We have studied the elliptic Gaudin models and solutions of the corresponding KZ equations. In the first part of the paper, we have constructed the Gaudin models based on the face-type elliptic quantum groups. Hamiltonians of the models are diagonalized by using the algebraic Bethe ansatz. With the help of these Gaudin model Hamiltonians, we have presented two types of elliptic KZ equations and give their solutions in the form of integrals. Then in the second part of the paper, we have constructed and diagonalized the boundary XYZ Gaudin model.

Acknowledgements:

This work has been financially supported by Australian Research Council. One of the authors (Zhao) would like to thank Professor Wei Wang for encouragement and the Mathematics Department of the University of Queensland for hospitality.

Note Added: After we finished this paper, we became aware that the integrable representation of solutions of the face-type elliptic KZ equations in the *periodic* case have been obtained in [24, 25, 26] by means of techniques different from the ones presented in this paper. We would like to thank T. Takebe for kindly drawing our attention to these references and the reference [21].

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